Note on a Micropolar Gas-Kinetic Theory

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Abstract. The micropolar fluid mechanics and its transport coefficients are derived from the linearized Boltzmann equation of rotating particles. In the dilute limit, as expected, transport coefficients relating to microrotation are not important, but the results are useful for the description of collisional granular flow on an inclined slope. (This paper will be published in Traffic and Granular Flows 2001 edited by Y. Sugiyama and D. E. Wolf(Springer)).

1 Introduction

Micropolar fluids are fluids with micro-structures. They belong to a class of fluid with a non-symmetric stress tensor. Micropolar fluids consist of rigid, randomly oriented (or spherical) particles which have own spins and microrotations suspended in a viscous medium. The concept of microrotation, originally, has been proposed by Cosserat brothers in the theory of elasticity[1]. Condiff and Dahler[2], and Eringen[3] applied its concept to describe fluids with micro-structures in the middle of 60s. Recently, a comprehensive textbook on micropolar fluids has been published[4].

Physical examples of micropolar fluids can be seen in many fields where all of them contain intrinsic polarities. However, we believe that the most interesting application of micropolar fluid dynamics which would have potential is to characterize granular flows[5,6,7]. In fact, the granular flow is one of flows which have micro-structure and rotation of particles. Therefore, it is natural to introduce a fluid model which contains an equation for microrotation in addition to an equation of velocity. Along this idea, Kanatani[8] formulated micropolar fluid mechanics for granular flows, which contains the angle of repose within the frame work. Kano *et al.*[9] have confirmed the quantitative validity of a micropolar fluid model in a chute flow of granular particles from the comparison of their simulation of micropolar fluids with their experience. It is worthwhile to introduce that the velocity profile obtained from the micropolar fluid model[9] is far from the parabolic curve expected from the conventional Novier-Stokes flow.

Although we do not know how the micropolar fluid model is relevant in other situations of granular materials, it is worthwhile to investigate fundamental properties of micropolar fluids from the view of granular physics (see e.g. [10]). Thus, the micropolar fluid mechanics is not a nonsense generalization of the Navier-Stokes model, but is a physically relevant model which has many applications in physical systems.

Recently, Mitarai *et al.*[11] have analyzed the collisonal flow of particles on an inclined slope. They suggested that the kinetic theory of granular particles is useful to derive micropolar fluid mechanics, and the fluid equation is quantitatively relevant to characterize the granular flow. Their success of the application of micropolar fluid mechanics to granular flows makes us understand microscopic origin of the micropolar fluid mechanics. Such attempts have been discussed within kinetic theory of polyatomic fluids [12,13] and there are some applications to granular fluid[14], but most of their works are not well accepted in these days.

The purpose of this paper is to summarize our current understanding of kinetic theory of micropolar gases. For the sake of simplicity we restrict our interest to two-dimensional perfectly rough particles in the dilute limit. To avoid complication which may be one of reasons for not appealing of the old kinetic theory of rough particles, we adopt the method of eigenvalue analysis of linearized Boltzmann equation.[15]

The organization of this paper is as follows. In section 2 we explain the outline of micropolar fluid mechanics and its eigenvalue analysis. In section 3, we shortly summarize the classical mechanics of binary collisions of identical rough disks. In section 4, we explain the framework of kinetic theory of dilute gases of rough disks. We also summarize the relation between the eigenvalues of linearized Boltzmann equation and the transport coefficients. In section 5, we demonstrate how to obtain transport coefficients in micropolar fluids. In section 6 we will discuss our results.

2 Outline of Micropolar Fluid Mechanics and its Eigenvalue Analysis

2.1 Outline of Micropolar Fluid Mechanics

In this section, we explain the outline of micropolar fluid mechanics. The conventional fluid mechanics consists of equations of continuity of mass, linear momentum and energy. In the fluid without microstructure the conservation of angular momentum is automatically satisfied but it becomes a nontrivial conservation law in the fluid with microstructure. Therefore we need an extra equation of angular momentum in micropolar fluid mechanics. In collisional flows we can adopt the differential expansion of the strain field in the stress tensor, the couple stress and the heat flux. The constitutive equation is similar to that in the Newtonian fluid. In this paper we restrict our interest to two dimensional flows.

The equation of mass conservation is the same as that in usual fluid

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1}$$

where $\partial_t = \partial/\partial t$, and ρ and $\mathbf{u} = (u_x, u_y, 0)$ are the mass density and the macroscopic velocity field, respectively.

The continuity equation of linear momentum, on the other hand, becomes

$$\rho D_t \mathbf{u} = \nabla \cdot \sigma \tag{2}$$

where $D_t = \partial_t + (\mathbf{u}\cdot\nabla)$ is Lagrange's derivative. Here the stress tensor σ contains the asymmetric part. In collisional flows in two dimension, the stress tensor σ_{ij} can be expanded by strains

$$\begin{aligned}\sigma_{ij} = &(-p + \zeta\partial_k u_k)\delta_{ij} + \mu(\partial_i u_j + \partial_j u_i - \partial_k u_k \delta_{ij}) \\ &+\mu_r(\partial_i u_j - \partial_j u_i - 2\epsilon_{kij} w_k),\end{aligned} \tag{3}$$

where ϵ_{kij} and $w_k = w\delta_{k,3}$ are respectively Eddigton's epsilon and the microrotation (macroscopic spin field). Thus, the micropolar fluid mechanics has the spin viscosity μ_r except for the usual viscosity μ and the bulk viscosity ζ.

The continuity equation of angular velocity is

$$nID_t w = \partial_j C_{j3} + \epsilon_{3ij}\sigma_{ij}, \tag{4}$$

where the component of the couple stress C_{ij} for two-dimensional micropolar flow is given by $C_{13} = \mu_B \partial_x w$, $C_{23} = \mu_B \partial_y w$ and $C_{33} = 0$. n and I are respectively the number density and the momentum of inertia of each disk. There is the relation between n and ρ as $\rho = nm$ with the mass of particle m.

The energy conservation law, in general, can be written as

$$\rho D_t e = -\partial_i q_i + \sigma_{ij}\partial_i u_j + C_{ij}\partial_i w_j - \epsilon_{kij}\sigma_{ij} w_k - \Gamma \tag{5}$$

where e, Γ and q_i are respectively the energy density, the dissipation rate by inelastic collisions and the heat flux. Now we may assume Fourier's law $q_i = -\kappa\partial_i T$ with the heat conductivity κ and the granular temperature. In general, the energy density of micropolar fluid consists of two parts, the translational energy and the rotational energy. However, the mechanism of the energy transfer between two parts is not simple[16]. Here, we are only interested in the total energy density of two parts. We also assume $\Gamma = 0$ to characterize of behavior of small dissipations. We will discuss the complete treatment including the energy transfer between two parts and the energy loss of inelastic collisions elsewhere.

2.2 Linearized hydrodynamics and its eigenvalue analysis

Here let us discuss the behavior of linearized hydrodynamics around the basic state $\bar\rho$, $\bar e$ and $\bar{\mathbf{U}} = 0$:

$$\rho = \bar\rho + \delta\rho, \quad \mathbf{U} = \delta\mathbf{U}, \quad e = \bar e + \delta e, \tag{6}$$

where $\mathbf{U} = (\mathbf{u}, w) = (u_x, u_y, w)$ is the three dimensional velocity-microrotation vector. Assuming the relation between δe and δT as $\delta e = C_V \delta T$ with the specific heat with constant volume C_V, the set of equations for linearized hydrodynamics in Fourier space thus becomes

$$\begin{aligned}\partial_t \rho_q &= -i\bar\rho\mathbf{q}\cdot\mathbf{u}_q \\ \partial_t \mathbf{u}_q &= -i\alpha\mathbf{q}\cdot\rho_q - i\beta\mathbf{q}\cdot T_q - \nu q^2\mathbf{u}_q - \delta\mathbf{q}(\mathbf{q}\cdot\mathbf{u}_q) + 2i\nu_r\mathbf{q}\times\mathbf{w}_q \\ \partial_t w_q &= -\hat\mu_B q^2 w_q + 2\hat\mu_r(i\mathbf{q}\times\mathbf{u}_q - 2w_q) \\ \partial_t T_q &= -i\gamma\mathbf{q}\cdot\mathbf{u}_q - \eta q^2 T_q\end{aligned} \tag{7}$$

where $\mathbf{w}_q = (0, 0, w_q)$ is the microrotation vector and the suffix q represents the Fourier component of the variable and

$$\begin{aligned}
\alpha &= \frac{1}{\rho}\left(\frac{\partial p}{\partial \rho}\right)_T, \quad \beta = \frac{1}{\rho}\left(\frac{\partial p}{\partial T}\right)_\rho, \quad \nu = \frac{\mu+\mu_r}{\rho} \\
\delta &= \frac{1}{\rho}(\zeta - \mu_r), \quad \gamma = \frac{T}{\rho C_V}\left(\frac{\partial p}{\partial T}\right)_\rho, \quad \eta = \frac{\kappa}{\rho C_V} \\
\hat{\mu}_B &= \frac{\mu_B}{nI}, \quad \hat{\mu}_r = \frac{\mu_r}{nI}, \quad \nu_r = \frac{\mu_r}{\rho}.
\end{aligned} \tag{8}$$

When we assume $\mathbf{q} = q\hat{x} = (q, 0, 0)$, the set of equations (7) can be summarized as

$$\partial_t \boldsymbol{\Psi}_q = M_q \cdot \boldsymbol{\Psi}_q, \tag{9}$$

where $\boldsymbol{\Psi}_q = {}^T(\rho_q, u_{x,q}, u_{y,q}, w_q, T_q)$ and

$$M_q = \begin{pmatrix} 0 & -iq\bar{\rho} & 0 & 0 & 0 \\ -i\alpha q & -(\nu+\delta)q^2 & 0 & 0 & -i\beta q \\ 0 & 0 & -\nu q^2 & -2i\nu_r q & 0 \\ 0 & 0 & -2i\hat{\mu}_r q & -\hat{\mu}_B q^2 - 4\hat{\mu}_r & 0 \\ 0 & -i\gamma q & 0 & 0 & -\eta q^2 \end{pmatrix}. \tag{10}$$

Therefore, the eigen equation can be written as

$$M_q \boldsymbol{\varphi}_\alpha^q = \lambda_\alpha^q \boldsymbol{\varphi}_\alpha^q. \tag{11}$$

With the aid of five eigenvectors φ_α^q, the solution of linearized hydrodynamics (9) is represented by

$$\boldsymbol{\Psi}_q(t) = \sum_{\alpha=1}^{5} c_\alpha^q(t) \boldsymbol{\varphi}_\alpha^q, \tag{12}$$

where $c_\alpha^q(t)$ behaves as $c_\alpha^q(t) = c_\alpha^q(0)e^{\lambda_\alpha^q t}$.

The eigenvalues of eq.(9) is connected with the transport coefficients, where the eigenvalues are the solution of

$$det\{M_q - \lambda_\alpha^q I\} = 0. \tag{13}$$

Although the exact solution of (13) is difficult to be obtained, the hydrodynamic solution near $q = 0$ can be obtained easily. Following the method introduced in standard textbooks (see e.g. [15]) , the five eigenvalues are obtained as follows: The first two eigenvalues are

$$\lambda_{1,2}^q = \pm i c_s q - \Gamma_s q^2, \tag{14}$$

where $c_s = \sqrt{(C_p/C_V)(\partial p/\partial \rho)_T}$ with the specific heat with constant pressure C_p. Γ_s is the rate of sound absorption, which is given by $\Gamma_s = \frac{1}{\bar{\rho}}\left(\zeta + \mu + \kappa(\frac{1}{C_V} - \frac{1}{C_p})\right)$. The third mode is

$$\lambda_3^q = -\frac{\mu}{\bar{\rho}} q^2, \tag{15}$$

and the fifth mode is

$$\lambda_5^q = -\frac{\kappa}{\bar{\rho}C_p}q^2. \tag{16}$$

The fourth mode is the nontrivial mode which represents the relaxation of the microrotation as

$$\lambda_4^q = -\frac{4\mu_r}{nI} - \frac{1}{nI}(\mu_B + \frac{\mu_r I}{m})q^2. \tag{17}$$

We should note that λ_4^q contains a constant which is independent of q. This means that the microrotation field cannot be connected with the zero eigenvectors of the linearized Boltzmann (or Enskog) equation.

3 Binary Collisions of Identical Rough Disks

In this section, we briefly summarize the result of binary collisions of two-dimensional, identical, circular disks obtained by Jenkins and Richman[17].

Here we assume that all variables are non-dimensionalized by the diameter of the disk d, the thermal velocity $c_T \equiv \sqrt{2T_0/m}$, where T_0 and m are granular temperature and the mass of a particle, respectively. The angular velocity is assumed to be non-dimensionalized by $\omega_T \equiv \sqrt{2T_0/I}$ with $I = md^2/8$. All dimensionless quantities are specified by hat. Thus, we introduce

$$\hat{x} = \frac{x}{d}, \quad \hat{\mathbf{c}} = \frac{\mathbf{c}}{c_T}, \quad \hat{\omega} = \frac{\omega}{\omega_T}. \tag{18}$$

which means that $d\omega$ is reduced to $2\sqrt{2}\hat{\omega}c_T$.

Let us consider a binary collision in which the translational velocity $\hat{\mathbf{c}}_1$, $\hat{\mathbf{c}}_2$ and the spin $\hat{\omega}_1$, $\hat{\omega}_2$ priori to a collision to become $\hat{\mathbf{c}}_1'$, $\hat{\mathbf{c}}_2'$ and $\hat{\omega}_1'$, $\hat{\omega}_2'$. In order to characterize binary collisions of two identical disks we introduce two coefficients of restitution as

$$\hat{k} \cdot \hat{\mathbf{v}}' = -e\hat{k} \cdot \hat{\mathbf{v}} \quad \text{and} \quad \hat{k} \times \hat{\mathbf{v}}' = -\beta_0(\hat{k} \times \hat{\mathbf{v}}) \tag{19}$$

where $\hat{k}$ is the unit vector connecting the center of the particle 1 with that of 2, and $\hat{\mathbf{v}}$ is the relative velocity of the points of contact defined by

$$\hat{\mathbf{v}} = \hat{\mathbf{g}} + \sqrt{2}\hat{z} \times \hat{k}(\omega_1 + \omega_2) \quad \text{with} \quad \hat{\mathbf{g}} = \hat{\mathbf{c}}_1 - \hat{\mathbf{c}}_2. \tag{20}$$

The coefficients of restitution are not constants in actual situation. In particular, the tangential restitution coefficient β_0 strongly depends on the ratio of the tangential velocity to the normal velocity when the Coulomb slip takes place. However, they may be regarded as constants in a wide range approximately. The coefficient e may take values from 0 to 1, while β_0 may vary between -1 and 1. The elastic hard core collision is characterized by $e = 1$ and $\beta_0 = -1$ but the collisions of perfectly rough disks are characterized by $e = \beta_0 = 1$, where the energy is conserved in the binary collisions and there is still the time reversal symmetry.

In this paper, we are interested in the case $e = \beta_0 = 1$, because this situation allows the simplified description with time reversal symmetry but the spin can play nontrivial roles. In this situation, we have the changes in a binary collision of the relative velocity $\hat{\mathbf{g}}$

$$\hat{\mathbf{g}}' - \hat{\mathbf{g}} = -\frac{2}{3}\hat{\mathbf{v}} - \frac{4}{3}\hat{k}(\hat{\mathbf{g}} \cdot \hat{k}), \tag{21}$$

the velocity

$$\hat{\mathbf{c}}_1' - \hat{\mathbf{c}}_1 = -\frac{\hat{\mathbf{v}}}{3} - \frac{2}{3}\hat{k}(\hat{k} \cdot \hat{\mathbf{g}}), \qquad \hat{\mathbf{c}}_2' - \hat{\mathbf{c}}_2 = \frac{\hat{\mathbf{v}}}{3} + \frac{2}{3}\hat{k}(\hat{k} \cdot \mathbf{g}), \tag{22}$$

and the spin

$$\hat{\omega}_1' - \hat{\omega}_1 = -\frac{\sqrt{2}}{3}(\hat{k} \times \hat{\mathbf{v}})\hat{z}, \qquad \hat{\omega}_2' - \hat{\omega}_2 = -\frac{\sqrt{2}}{3}(\hat{k} \times \hat{\mathbf{v}})\hat{z}. \tag{23}$$

More general results for any e and β_0 can be seen in the paper by Jenkins and Richman[17]. For example, let us present one result for any β_0 as

$$\hat{\omega}_1^* + \hat{\omega}_2^* - \hat{\omega}_1 - \hat{\omega}_2 = -\frac{\sqrt{2}}{3\beta_0}(1 + \beta_0)\hat{z} \cdot (\hat{k} \times \hat{\mathbf{v}}), \tag{24}$$

where $\hat{\omega}_i^*$ denotes the spin of i priori to a collision to become $\hat{\omega}_i$. Note that there is time reversal symmetry for any e and β_0.

4 Kinetic Theory of Dilute Gases

In this section, we discuss the kinetic theory of dilute rough disks. Since the Boltzmann equation should have modification if there is no time reversal symmetry, here we assume that the disks are perfectly rough, i.e. $e = \beta_0 = 1$ to keep the time reversal symmetry. As will be shown, the result recovers usual Navier-Stokes equation in the dilute limit, but in some situation like in flows on an inclined rough plate the prediction of the dilute gas kinetics seems to be quantitatively useful[11]. The extension of this analysis for dense particles with dissipative collisions will be discussed elsewhere. Note that there are some treatments of three dimensional dissipation-less polyatomic fluids[12,13] based on Chapman-Enskog scheme. On the other hand, Jenkins and Richman[17] developed kinetic theory of rough inelastic disks in two dimensions based on the Grad expansion. It is difficult to check their argument because their calculation is long and complicated. In addition, their method cannot derive micropolar fluid mechanics as a closed form. Here we introduce a simpler method of calculation which can derive micropolar fluid mechanics and can determine the transport coefficients.

We begin with Boltzmann equation of particles with diameter d:

$$\partial_t f_1 + (\mathbf{c}_1 \cdot \nabla) f_1 = d \int d\mathbf{c}_2 \int_{-\infty}^{\infty} d\omega_2 \int d\hat{k} H(\hat{\mathbf{g}} \cdot \hat{k}) g \{ f_1' f_2' - f_1 f_2 \} \tag{25}$$

Here we consider the collisions $(\mathbf{c}^*, \omega^*) \to (\mathbf{c}, \omega) \to (\mathbf{c}', \omega')$, but we note $(\mathbf{c}^*, \omega^*) = (\mathbf{c}', \omega')$ because of time reversal symmetry for $e = \beta_0 = 1$. In eq.(25), we use the following notations: $\hat{k}$ is the common unit normal vector at contact, $\hat{\mathbf{g}} = \hat{\mathbf{c}}_1 - \hat{\mathbf{c}}_2$, $\hat{g} = |\hat{\mathbf{g}}|$, $H(x) = 1$ for $x \geq 0$ and $H(x) = 0$ for $x < 0$, $f_1' \equiv f(\mathbf{r}, \mathbf{c}_1', \omega_1')$ and $f_1 = f(\mathbf{r}, \mathbf{c}_1, \omega_1)$.

Now let us analyze a linearly nonequilibrium situation. The distribution function can be expanded as

$$f(\mathbf{r}, \mathbf{c}, \omega, t) = n(\mathbf{r}, t) f_0(\mathbf{c}, \omega)(1 + \Psi(\mathbf{r}, \mathbf{c}, \omega)) \tag{26}$$

where n is the number density, and f_0 is the Maxwell-Boltzmann distribution function which vanishes in the collisional integral in eq.(25).

Now we adopt dimensionless quantities as in the previous section. Here we only introduce two variables as

$$\hat{n}(\hat{x}) = n(x) d^2, \quad f_0 = \frac{m\sqrt{I}}{(2T_0)^{3/2}} M(\xi) \tag{27}$$

where $\boldsymbol{\xi} = (\mathbf{c}, \hat{\omega})$ and $M(\boldsymbol{\xi}) = \exp(-\boldsymbol{\xi}^2)/\pi^{3/2}$.

It is easy to show that the zeroth order equation is given by $d\hat{n}/d\hat{x} = 0$. Thus, the distribution function f depends on the spatial coordinate only through Ψ.

The perturbative equation is now reduced to

$$\partial_t \Psi_1 + (\hat{\mathbf{c}}_1 \cdot \hat{\nabla}) \Psi_1 = \hat{n} \int d^3 \boldsymbol{\xi} \int d\hat{k} H(\hat{\mathbf{g}} \cdot \hat{k}) \hat{g} M(\boldsymbol{\xi}) \{ \Psi_2' + \Psi_1' - \Psi_2 - \Psi_1 \} \tag{28}$$

This equation can be written as

$$\partial_t \Psi_1 + (\hat{\mathbf{c}}_1 \cdot \hat{\nabla}) \Psi_1 = \hat{n} L[\Psi_1], \tag{29}$$

where $L[\Psi_1]$ is the collisional integral in eq.(28). Introducing Fourier transform we may rewrite (29) as

$$\partial_t \Psi_q + iq\hat{c}_x \Psi_q = \hat{n} L[\Psi_q]. \tag{30}$$

The solution of (30) can be obtained from the non-Hermitian eigenvalue problem

$$(\hat{L} - iq\hat{c}_x)|\Psi_j^q> = \lambda_j^q |\Psi_j^q> . \tag{31}$$

Since we are interested in hydrodynamic behavior of eq.(31), we adopt the expansion around $q = 0$ as

$$\begin{aligned} |\Psi_j^q> &= |\Psi_j^{(0)}> + q|\Psi_j^{(1)}> + q^2 |\Psi_j^{(2)}> + \cdots \\ \lambda_j^q &= \lambda_j^{(0)} + q\lambda_j^{(1)} + q^2 \lambda_j^{(2)} + \cdots . \end{aligned} \tag{32}$$

Substituting (32) into (30) we obtain

$$\hat{n} L |\Psi_j^{(0)}> = \lambda_j^{(0)} |\Psi_j^{(0)}> . \tag{33}$$

Thus, $|\Psi_j^{(0)}>$ is represented by the linear combination of five fundamental eigenvectors as:

$$|\Psi_\alpha^{(0)}>=\sum_{\alpha'=1}^{5} c_{\alpha\alpha'}|\phi_{\alpha'}>, \tag{34}$$

where $|\phi_\alpha>$ satisfies $\hat{L}|\phi_\alpha>=\lambda_\alpha^{(0)}|\phi_\alpha>$, and their explicit expressions are

$$\begin{aligned}&|\phi_1>=1, \quad |\phi_2>=\hat{c}_x, \quad |\phi_3>=\hat{c}_y,\\ &|\phi_4>=\hat{\omega}, \quad |\phi_5>=\sqrt{\frac{2}{3}}(\boldsymbol{\xi}^2-\frac{2}{3}).\end{aligned} \tag{35}$$

Since $|\phi_\alpha>$ is the degenerated eigenvector, we need to use $|\Psi_\alpha^{(0)}>$. The determination of $|\Psi_j^{(0)}>$ will be discussed later. We also assume that the eigenfunctions are orthonormal as

$$<\Psi_i^{(0)}|\Psi_j^{(0)}>\equiv\int d\boldsymbol{\xi}\Psi_j{}^{(0)}\Psi_i{}^{(0)}M(\boldsymbol{\xi})=\delta_{ij} \tag{36}$$

Therefore we may introduce $\bar{\lambda}_j^{(0)}$ as

$$\bar{\lambda}_j^{(0)}=\hat{n}<\Psi_j^{(0)}|L|\Psi_j^{(0)}>. \tag{37}$$

This $\bar{\lambda}_j^{(0)}$ is equivalent to $\lambda_j^{(0)}$ if $\lambda_j^{(0)}$ is independent of $\boldsymbol{\xi}$. If $\lambda_j^{(0)}$ is a function of $\boldsymbol{\xi}$, two eigenvalues are different from each other. We believe that $\bar{\lambda}_j^{(0)}$ plays fundamental roles in later discussion.

With the aid of (36) and (37) we obtain the relations at the first order:

$$|\Psi_j^{(1)}>=\frac{iq\hat{c}_x+\lambda_j^{(1)}}{\hat{n}L-\lambda_j^{(0)}}|\Psi_j^{(0)}> \tag{38}$$

and

$$\bar{\lambda}_j^{(1)}=i<\Psi_j^{(0)}|q\hat{c}_x|\Psi_j^{(0)}>, \tag{39}$$

where we use $L=L^\dagger$ and $\hat{n}<\Psi_4^{(0)}|L=\lambda_4^{(0)}<\Psi_4^{(0)}|$.

From (31), (32) and (38) we obtain the second order correction of the eigenvalue as

$$\lambda_j^{(2)}=-<\Psi_j^{(0)}|(iq\hat{c}_x+\lambda_j^{(1)})\frac{1}{\hat{n}L-\lambda_j^{(0)}}(iq\hat{c}_x+\lambda_j^{(1)})|\Psi_j^{(0)}>. \tag{40}$$

As the eigenvalues $\lambda_\alpha^0=0$ are degenerate, we must be careful in starting from a proper basis that avoids the appearance of vanishing denominators. As in the case of quantum mechanics, we must solve exactly eigenvalue problem in the subspace spanned by $|\phi_\alpha>$. From the comparison of (38) with (34) we obtain

$$\sum_{\alpha'=1}^{5} c_{\alpha\alpha'}[<\phi_{\alpha'}|\hat{c}_x|\phi_{\alpha''}>-\tilde{\lambda}_\alpha\delta_{\alpha,\alpha'}]=0, \tag{41}$$

where $\tilde{\lambda} = i\lambda_\alpha^{(1)}/q$. The required condition to exist nontrivial solutions of this equation is

$$det[< \phi_\beta | c_x | \phi_\alpha > - \tilde{\lambda}\delta_{\alpha\beta}] = 0 \tag{42}$$

Thus, we obtain

$$\tilde{\lambda}_1 = -\tilde{\lambda}_2 = \frac{1}{2}\sqrt{\frac{5}{3}}, \quad \tilde{\lambda}_3 = \tilde{\lambda}_4 = \tilde{\lambda}_5 = 0. \tag{43}$$

For later convenience, we introduce $c_0 \equiv \tilde{\lambda}_1$. Coming back to eq.(41) we obtain

$$\begin{aligned}
|\Psi_1^{(0)} > &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{3}{5}}|\phi_1 > + |\phi_2 > + \sqrt{\frac{2}{5}}|\phi_5 >\right], \\
|\Psi_2^{(0)} > &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{3}{5}}|\phi_1 > - |\phi_2 > + \sqrt{\frac{2}{5}}|\phi_5 >\right], \\
|\Psi_3^{(0)} > &= |\phi_3 >, \\
|\Psi_4^{(0)} > &= |\phi_4 >, \\
|\Psi_5^{(0)} > &= \sqrt{\frac{2}{5}}\left[-|\phi_1 > + \sqrt{\frac{3}{2}}|\phi_5 >\right].
\end{aligned} \tag{44}$$

These $|\Psi_\alpha^{(0)} >$ will be used as the basis of perturbative treatment.

Taking into account (40) we obtain the eigenvalues up to the second order as

$$\begin{aligned}
\lambda_1 &= -ic_0 q + q^2 < \Psi_1^{(0)} | (\hat{c}_x - c_0)\frac{1}{\hat{n}L}(\hat{c}_x - c_0) | \Psi_1^{(0)} >, \\
\lambda_2 &= ic_0 q + q^2 < \Psi_2^{(0)} | (\hat{c}_x + c_0)\frac{1}{\hat{n}L}(\hat{c}_x + c_0) | \Psi_2^{(0)} >, \\
\lambda_3 &= q^2 < \Psi_3^{(0)} | \hat{c}_x \frac{1}{\hat{n}L}\hat{c}_x | \Psi_3^{(0)} >, \\
\lambda_4 &= \lambda_4^{(0)} + q^2 < \Psi_4^{(0)} | \hat{c}_x \frac{1}{\hat{n}L}\hat{c}_x | \Psi_4^{(0)} >, \\
\lambda_5 &= q^2 < \Psi_5^{(0)} | \hat{c}_x \frac{1}{\hat{n}L}\hat{c}_x | \Psi_5^{(0)} > .
\end{aligned} \tag{45}$$

Basically, the eigenvalues obtained in this section should be equivalent to those obtained in section 2 as the linear micropolar hydrodynamics. We have to note, however, that the results in this section is written in dimensionless forms but the results in section 2 include physical dimensions in terms of the thermal velocity $c_T = \sqrt{2T_0/m}$ and the diameter of particles d. Thus we obtain the relations

$$\begin{aligned}
c_s \ &= c_0\sqrt{\frac{2T}{m}}, \\
\Gamma_s \ &= d\sqrt{\frac{2T_0}{m}} < \Psi_1^{(0)} | (\hat{c}_x - c_0)\frac{1}{\hat{n}L}(\hat{c}_x - c_0) | \Psi_1^{(0)}, >
\end{aligned}$$

$$
\begin{aligned}
\mu &= \rho d \sqrt{\frac{2T_0}{m}} < \Psi_3^{(0)} | \hat{c}_x \frac{1}{\hat{n}L} \hat{c}_x | \Psi_3^{(0)} > \\
\mu_r &= -\frac{nI}{4d} \sqrt{\frac{2T_0}{m}} \lambda_4^{(0)}, \\
\mu_B &= -\mu_r \frac{I}{m} - dnI \sqrt{\frac{2T_0}{m}} < \Psi_4^{(0)} | \hat{c}_x \frac{1}{\hat{n}L} \hat{c}_x | \Psi_4^{(0)} > \\
\kappa &= -\rho d C_p \sqrt{\frac{2T}{m}} < \Psi_5^{(0)} | \hat{c}_x \frac{1}{\hat{n}L} \hat{c}_x | \Psi_5^{(0)} > .
\end{aligned}
\tag{46}
$$

5 Evaluation of transport coefficients

In this section, we evaluate the transport coefficients μ, μ_r and μ_B among many transport coefficients. Actually we can describe incompressible micropolar fluid mechanics in terms of these three coefficients.

5.1 Evaluation of μ_r

As was shown in the previous section, to evaluate μ_r we have to obtain $\lambda_4^{(0)}$. We here present the result for any β_0, since it is possible to obtain $\lambda_4^{(0)}$ for any β_0 in (19).

Substituting (24) into (28) we obtain

$$
L[\hat{\omega}] = -\frac{\sqrt{2}(1+\beta_0)}{3\beta_0} \{ \mathcal{L}_1[\hat{\omega}] + \mathcal{L}_2[\hat{\omega}] \} = \frac{\lambda_4^{(0)}}{\hat{n}} \hat{\omega}, \tag{47}
$$

where

$$
\mathcal{L}_1[\hat{\omega}] = -\int d^2\hat{\mathbf{c}}_2 \int d\hat{k} H(\hat{\mathbf{g}} \cdot \hat{k})(\hat{\mathbf{g}} \cdot k) M_2(\hat{\mathbf{c}}) \hat{z} \cdot (\hat{k} \times \hat{\mathbf{g}}) \tag{48}
$$

and

$$
\mathcal{L}_2[\hat{\omega}] = \sqrt{2}\hat{\omega} \int d^2\hat{\mathbf{c}}_2 \int d\hat{k} H(\hat{\mathbf{g}} \cdot \hat{k})(\hat{\mathbf{g}} \cdot \hat{k}) M_2(\hat{\mathbf{c}}) \tag{49}
$$

with $M_2(\hat{\mathbf{c}}) = \exp[-\hat{c}_x^2 - \hat{c}_y^2]/\pi$. With the aid of formulae for the modified Bessel function $I_0(z)$ with zeroth order and the confluent Hypergeometric function $F(a, b; z)$:

$$
I_0(z) = \frac{1}{\pi} \int_0^{\pi} dx e^{z \cos x}, \quad F(3/2, 1; c^2) = \frac{4}{\sqrt{\pi}} \int_0^{\infty} dr r^2 I_0(2rc) \tag{50}
$$

it is possible to show

$$
\mathcal{L}_2[\hat{\omega}] = \frac{\sqrt{2\pi}}{2} F(3/2, 1; \hat{c}^2) e^{-\hat{c}^2} \hat{\omega}, \quad \text{and} \quad \mathcal{L}_1[\hat{\omega}] = 0. \tag{51}
$$

Therefore, we obtain the eigenvalue

$$
\lambda_4^{(0)} = -\frac{\sqrt{\pi}(1+\beta_0)\hat{n}}{3\beta_0} e^{-\hat{c}^2} F(3/2, 1; \hat{c}^2). \tag{52}
$$

This result states that the eigenvalue depends on the velocity of particles.

As mentioned in section 4, if the eigenvalue depends on the velocity, the effective eigenvalue $\bar{\lambda}_\alpha$ defined in (37) plays important roles. Thus, $\bar{\lambda}_4^{(0)}$ is evaluated as

$$\bar{\lambda}_4^{(0)} = -\frac{\sqrt{\pi}(1+\beta_0)\hat{n}}{3\beta_0}\int d^3\boldsymbol{\xi}\frac{e^{-\boldsymbol{\xi}^2}}{\pi^{3/2}}e^{-\hat{c}^2}F(3/2,1;\hat{c}^2)\hat{\omega}^2. \quad (53)$$

The result of integration becomes

$$\bar{\lambda}_4^{(0)} = -\frac{\sqrt{2\pi}(1+\beta_0)}{6\beta_0}\hat{n} \quad (54)$$

where we use $\int_0^\infty dcce^{-2c^2}F(3/2,1;c^2) = 1/\sqrt{2}$. We note that the result (54) is reduced to

$$\bar{\lambda}_4^{(0)} = -\frac{\sqrt{2\pi}}{3}\hat{n} \quad (55)$$

in the case of $\beta_0 = 1$.

From (45), (46) and (55) we obtain

$$\mu_r = \frac{\sqrt{2\pi}n^2}{96}md^3\sqrt{\frac{2T_0}{m}} \quad (56)$$

for $e = \beta_0 = 1$. Thus, μ_r is proportional to n^2 as predicted by Mitarai et al.[11]. Therefore, the effect of microrotation is negligible in usual situations of dilute gases. However, as demonstrated by Mitarai *et al.*[11] the shear near the boundary produces the relevant situation for microrotation.

5.2 Evaluation of $\boldsymbol{\mu}$

μ can be evaluated from the third equation of (46). This equation may be rewritten as

$$\hat{\mu} = -\int d\boldsymbol{\xi}\hat{c}_x\hat{c}_y\chi^{xy}M(\boldsymbol{\xi}) = -<\chi^{xy}|Y^{xy}>, \quad (57)$$

where $|\chi^{xy}> = \chi^{xy}$ and $|Y^{xy}> = \hat{c}_x\hat{c}_y$. $\hat{\mu}$ is defined by

$$\hat{\mu} = \mu/(\rho d\sqrt{\frac{2T_0}{m}}) \quad (58)$$

and χ^{xy} is the solution of

$$\hat{n}L[\chi^{xy}] = \hat{c}_x\hat{c}_y \quad \text{or} \quad \hat{n}L|\chi^{xy}> = |Y^{xy}> . \quad (59)$$

Introducing the function $|f^{xy}>$ vertical to $|\phi_\alpha>$, it is easy to show that $<\bar{f}|\hat{L}|\bar{f}>$ is minimum for $|\bar{f}> = |\chi^{xy}>$ where $|\bar{f}> = <f^{xy}|Y^{xy}> / <f^{xy}|\hat{n}L|f^{xy}> |f^{xy}>$. The minimum principle then becomes

$$<\chi|Y^{xy}> \le <\bar{f}|\hat{n}L|\bar{f}> = \frac{|<f^{xy}|Y^{xy}>|^2}{<f^{xy}|\hat{n}L|f^{xy}>}. \quad (60)$$

Here we assume the following expansion:

$$|f^{xy}>=\sum_{j=1}^{m}\beta_j^{(m)}|f_j>, \tag{61}$$

where the coefficient $\beta_j^{(m)}$ is determined by the minimum condition of the right hand side of (60). These conditions are summarized as

$$Y_j^{xy}=\sum_{l=1}\beta_l^{(m)}b_{jl}^{xy}, \quad <\chi^{xy}|Y^{xy}>=-\frac{1}{\hat{n}}\sum_{j=1}^{m}\beta_j^{(m)}Y_j^{xy}, \tag{62}$$

where $Y_k^{xy}=<f_k|Y^{xy}>$ and $b_{kl}^{xy}=-<f_k|L|f_l>$.

We assume that $|f_j>$ can be represented by

$$|f_j>=S_2^{(j-1)}(\hat{c}^2)\hat{c}_x\hat{c}_y, \tag{63}$$

where $S_2^{(j-1)}(x)$ is the Sonine polynomial which is defined by

$$S_l^{(r)}(x)=\sum_{j=0}^{r}\frac{(-1)^j\Gamma(l+r+1)}{\Gamma(l+j+1)(r-j)!j!}x^j \tag{64}$$

with the Gamma function $\Gamma(x)$. The Sonine polynomials satisfy the orthonormality condition $\int_0^\infty dx x^l e^{-x}S_l^{(r)}(x)S_l^{(r')}(x)=\frac{\Gamma(r+l+1)}{r!}\delta_{r,r'}$. In particular, the relations $S_l^{(0)}(x)=1$ is useful for later discussion. From the orthonomality the condition of $<f^{xy}|\phi_\alpha>=\sum_j^m\beta_j^{(m)}<f_j|\phi_\alpha>=0$ is automatically satisfied.

Therefore we obtain the expressions for Y_j^{xy} and b_{kl}^{xy} as

$$\begin{aligned} Y_j^{xy} &= \int d\boldsymbol{\xi}\hat{c}_x\hat{c}_y S_2^{(j-1)}(\hat{c}^2)\hat{c}_x\hat{c}_y M(\boldsymbol{\xi}), \\ b_{kl}^{xy} &= -\int d\boldsymbol{\xi} M(\boldsymbol{\xi})S_2^{(k-1)}(\hat{c}^2)\hat{c}_x\hat{c}_y L\hat{c}_x\hat{c}_y S_2^{(l-1)}(\hat{c}^2). \end{aligned} \tag{65}$$

It is easy to calculate Y_j as

$$Y_j^{xy}=\frac{1}{4}\delta_{j,1}. \tag{66}$$

It is notable that Y_k^{xy} is zero except for Y_1^{xy}.

Since it is known that the expansion in terms of Sonnine's polynomial is fast, the result with $m=1$ gives a good approximation. Adopting this approximation (62) becomes

$$<\chi^{xy}|Y^{xy}>=-\frac{\beta_1^{(1)}}{\hat{n}}Y_1^{xy}; \quad Y_1^{xy}=\beta_1^{(1)}b_{11}^{xy}. \tag{67}$$

Eliminating $\beta_1^{(1)}$ from (67) we obtain

$$<\chi^{xy}|Y^{xy}>=-\frac{Y_1^{xy\,2}}{\hat{n}b_{11}^{xy}}. \tag{68}$$

From the definition of the operator L b_{11}^{xy} can be written as

$$b_{11}^{xy} = \frac{1}{4\pi^3}\int d\boldsymbol{\xi}_1 \int d\boldsymbol{\xi}_2 \int_{-\pi/2}^{\pi/2} d\psi \cos\psi \hat{g} e^{-\boldsymbol{\xi}_1^2 - \boldsymbol{\xi}_2^2}$$
$$\times[\hat{c}'_{1x}\hat{c}'_{1y} + \hat{c}'_{2x}\hat{c}'_{2y} - \hat{c}_{1x}\hat{c}_{1y} - \hat{c}_{2x}\hat{c}_{2y}]^2. \qquad (69)$$

where $\pi - \psi$ is the angle between $\hat{\mathbf{g}}$ and $\hat{k}$. To derive (69) we have used that the cross section ($da/d\psi$ with the impact parameter a) is $\cos\psi$ and the time reversal symmetry of collisions.

From (21) and (69) we get the relation

$$b_{11}^{xy} = \frac{7}{18}\sqrt{\frac{\pi}{2}}. \qquad (70)$$

From (57), (61) and (68) we obtain

$$\hat{\mu} = \frac{9}{56\hat{n}}\sqrt{\frac{2}{\pi}}. \qquad (71)$$

From comparison of (46) with the definition of $\hat{\mu}$, the final expression of μ is given by

$$\mu = \frac{9}{28d}\sqrt{\frac{mT_0}{\pi}}. \qquad (72)$$

Note that this result is deviated from the result in ref.[17].

5.3 Evaluation of μ_B

The method of calculation of μ_B is similar to that of μ. Let us introduce

$$\mu_c = -\int d\boldsymbol{\xi}\hat{\omega}\hat{c}_x \chi^{\omega x} M(\boldsymbol{\xi}) = - < \chi^{\omega x}|Y^{\omega x} >, \qquad (73)$$

where $|Y^{\omega x} >= \hat{\omega}\hat{c}_x$ and $|\chi^{\omega x} >= \chi^{\omega x}$ is the solution of

$$\hat{n}L|\chi^{\omega x} >= \hat{|}Y^{\omega x} > . \qquad (74)$$

Similar to the previous section $\hat{\mu}_c$ is represented by

$$\mu_c = \frac{Y_1^{\omega x 2}}{\hat{n} b_{11}^{\omega x}} \qquad (75)$$

in the lowest order approximation. Here $Y_1^{\omega x}$ and $b_{11}^{\omega x}$ are respectively given by

$$Y_1^{\omega x} = \int d\boldsymbol{\xi}(\hat{\omega}\hat{c}_x)^2 M(\boldsymbol{\xi}), \quad b_{11}^{\omega x} = -\int d\boldsymbol{\xi} M(\boldsymbol{\xi})\hat{\omega}\hat{c}_x L\hat{\omega}\hat{c}_x. \qquad (76)$$

It is easy to show

$$Y_1^{\omega x} = \frac{1}{4}. \qquad (77)$$

On the other hand, $b_{11}^{\omega x}$ is rewritten as

$$b_{11}^{\omega x} = \frac{1}{4\pi^3}\int d\boldsymbol{\xi}_1 \int d\boldsymbol{\xi}_2 \int_{-\pi/2}^{\pi/2} d\psi \cos\psi \hat{g} e^{-\boldsymbol{\xi}_1^2-\boldsymbol{\xi}_2^2} \times[\hat{c}'_{1x}\hat{\omega}'_1 + \hat{c}'_{2x}\hat{\omega}'_2 - \hat{c}_{1x}\hat{\omega}_1 - \hat{c}_{2x}\hat{\omega}_2]^2. \tag{78}$$

From (22) and (23) we obtain

$$b_{11}^{\omega x} = \frac{11}{12}\sqrt{\frac{\pi}{2}}. \tag{79}$$

Thus μ_c becomes

$$\mu_c = \frac{3}{44}\sqrt{\frac{2}{\pi}}. \tag{80}$$

From (46) and (73) we obtain

$$\mu_B = \frac{3}{176\sqrt{\pi}} d\sqrt{mT_0} - \frac{\sqrt{2\pi}(nd^2)^2}{768} d\sqrt{2mT_0}. \tag{81}$$

It is obvious that the second term is negligible in the dilute limit.

The method of derivation of μ_B is relatively simple, when we compare the method based on Chapman-Enskog scheme in which μ_B becomes the correction term of higher order[12].

6 Discussion

Here, we have demonstrated how micropolar fluid mechanics can be derived from the Boltzmann equation. Of course, the result is reduced to Navier-Stokes equation in the dilute limit. In this sense, at least, we will have to extend our work to the case of Enskog equation if we believe that the concept of micropolar fluid mechanics is useful. The effect of dissipation also plays important roles though we do not discuss it.

Roughly speaking the effect of microrotation is localized in the boundary layer. Therefore it will be important to analyze simple shear flows. Our preliminary result suggests that the micropolar fluid mechanics may not be enough to discuss the region close to a flat boundary. The success by Mitarai *et al.*[11] may come from their bumpy boundary condition in which there is no Knudsen's layer in the system due to random scattering of particles near the boundary.

There are many problems to be solved. Let us close the paper with our perspective whether micropolar fluid mechanics is useful. In the pessimistic view the analysis presented here is general nonsense, and micropolar fluid mechanics is useless. On the other hand, in the optimistic view, this work will be a milestone to discuss the fluid motion with microstructure. Although we are not sure which result we will see in the future, we hope that micropolar fluid mechanics is a useful concept to characterize the flow of particles.

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